Instituto Superior das Ciências do Trabalho e da Empresa
Departamento de Finanças

Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

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Pricing of a Credit Default Swap

Sara Maria Correia Pereira

Dissertação
Mestrado em Matemática Financeira

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Orientador: Professor Doutor José Carlos Dias

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Acknowledgments

Para começar gostava de agradecer ao meu orientador, porque durante este trabalho soube guiar-me pelo caminho correcto, e fez o que podia para me orientar neste trabalho. Gostava de também agradecer tanto à Faculdade de Ciências da Universidade de Lisboa como também ao ISCTE pelos conhecimentos transmitidos pelos professores de ambas as faculdades durante o mestrado. Finalmente, agradecer à minha família que tanto me ajudou, psicologicamente, durante estes meses de escrita.

Abstract

The present work is dedicated to evaluate a Credit Default Swap, using as reference the model implemented by O'Kane and Turnbull in their paper *Valuation of Credit Default Swaps* (2003)[11]. For such valuation it is necessary to have acknowledgments about stochastic calculus as well as, some acknowledgments about how the world of the financial derivates instruments works. In this context, for the valuation of this credit derivative, it is necessary to study the default probability of both parties, and those probabilities need to be arbitrage free. So this model has to be able to capture the survival probability with the information given by the market of both parties. This model uses a reduced-form approach for credit modeling. The most used in this context, is based on the work presented in the paper of Jarrow and Turbull (1995)[7], where the authors characterize a credit event as the first event of a Poisson Counting Process. This event, occurs at some moment τ with a probability defined as $\Pr\left[\tau < \mathrm{t} + \mathrm{dt}\right] = \lambda(t)dt$, i.e, the probability of occurring some default in the time interval $[\mathrm{t}, \mathrm{t} + \mathrm{dt}]$ conditional to surviving up to t is proportional to some dependent function of t, known as hazard rate, and to the length of the time interval dt. In this work we assume a constant hazard rate in the survival time of the issuer. We also assume a recovery rate, R, for the bond that is linked to the credit default swap (CDS) and what the model implies is, in the case there is no default, the survival probability is $1 - \lambda(t)\mathrm{dt}$ and in the case there is a default, we receive a recovery rate R, with probability of $\lambda(t)\mathrm{dt}$.

For the implementation of a good model it is necessary to use a bootstrapping algorithm for the computing of the survival probability for each year of the CDS (this algorithm where implemented using *Matlab*). The result was: while the survival probability was decreasing, the hazard rate for each year where increasing, so for long maturities of the CDS the greater is the hazard rate and for consequence a higher default probability. So in the markets, the CDS's more traded is the one's with shortest maturities, such as 5 and 10 years.

For conclusion, this model is compared to the model implemented by the authors John Hull and Alan White in the paper *Valuing Credit Default Swaps I: No counterparty Default Risk*[6], using a 5 year CDS and a bond, both issued by Banco Espírito Santo in the time of their restructuring.

Keywords: Credit Default Swap, Default Probability, Hazard Rate, Recovery Rate.

Resumo

O presente trabalho é dedicado a avaliar um Credit Default Swap usando como referência o modelo implementado por O'Kane e Turnbull no paper *Valuation of Credit Default Swaps* (2003)[11]. Para essa avaliação é necessário ter conhecimentos de cálculo estocástico assim como alguns conhecimentos acerca de como o mundo dos instrumentos derivados funciona. Neste mesmo âmbito, para a avaliação deste derivado, é necessário estudar a probabilidade de default de ambas as contrapartes e estas probabilidades têm que ser livres de arbitragem. Portanto este modelo tem que ser capaz de capturar a probabilidade de sobrevivência a partir da informação que o mercado dispõe sobre ambas as contrapartes. Neste modelo considera-se um modelo de avaliação de crédito usando uma *reduced-form approach*. O mais usado neste âmbito é baseado no trabalho de Jarrow and Tunbull (1995)[7], onde os autores caracterizam um evento de crédito como o primeiro evento de um Processo de Contagem de Poisson. Este mesmo evento, ocorre em algum instante τ e com uma probabilidade definida como $\Pr\left[\tau < \mathrm{t} + \mathrm{dt}\right] = \lambda(t)dt$, ou seja a probabilidade de ocorrer algum *default* no intervalo de tempo $[\mathrm{t}, \mathrm{t} + \mathrm{dt}]$ condicional a ter sobrevivido até t é proporcional a alguma função dependente de t, conhecida como *hazard rate* , e ao comprimento do intervalo dt. Neste trabalho assumimos que a *hazard rate* é constante ao longo do tempo de sobrevivência do emitente. Assumimos também uma *recovery rate*, R, para a obrigação associada ao *credit default swap* (CDS) e o que o modelo implica é, caso a obrigação não entre em *default* a probabilidade de sobrevivência é $1 - \lambda(t)\mathrm{dt}$ e caso ocorra *default* recebe-se a *recovery rate* R, com probabilidade $\lambda(t)\mathrm{dt}$.

Para a implementação de um bom modelo de avaliação é necessário usar um algoritmo de *bootstrapping* para calcular a probabilidade de sobrevivência a cada ano do CDS (este mesmo algoritmo foi implementado usando o *Matlab*). O resultado que se obteve foi que: enquanto que a probabilidade de sobrevivência ia diminuindo a *hazard rate* encontrada ia aumentando, por isso quanto mais longo o CDS, maior seria a *hazard rate* e por consequência uma maior probabilidade de *default*. Por isso no mercado, os CDS's mais transaccionados são aqueles com maturidades mais curtas, como 5 ou 10 anos.

Para concluir, este modelo é também comparado com o modelo implementado pelos autores John Hull e Alan White no paper *Valuing Credit Default Swaps I: No counterparty Default Risk*[6], usando um CDS a 5 anos e uma obrigação, ambos emitidos pelo Banco Espírito Santo na altura da sua reestruturação.

Palavras-Chave: Credit Default Swap, Probabilidade de Default, *Hazard Rate*, *Recovery Rate*.

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Introduction

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1.1 An Introduction to Credit Default Swaps

"The cost of "credit default swaps" (CDS), which cover the credit risk of the Portuguese debt for five years, rose to the record level of 3.95 million in signing contracts, over 100 thousand dollars a year to renew contracts, according to data from CMA Datavision cited by Bloomberg."

Before initiating the valuation of a credit default swap, let us first explain the main characteristics of such instrument and present its applications in the derivative and financial markets.

1.1.1 A Swap Contract

A contract swap is an agreement between two counterparties to exchange cash-flows in the future. These cash flows usually are calculated based on some future value of an interest rate, an exchange rate or other market variable. These types of contract can be viewed as a forward contract, however unlike forward contracts, swaps typically lead to cash flows exchanges in several future dates.(Hull[5]) The most typically swap contract is a plain vanilla interest rate swap, and for a simpler brief we will explain how it works. So, an interest rate swap is an agreement between two parties to exchange a series of interest payments, but without exchanging the underlying debt. There are two kinds of rates, the fixed and floating rates, so in a typical rate swap, one party (known as fixed-rate payer) promises to pay to the second at designated intervals a stipulated amount of interests calculated at a fixed rate on the notional; and the other party (know as the floating rate payer) promises to pay to the first at the same intervals a floating amount of interest on the notional calculated according to a floating-rate index.(Bicksler and Chen[1]; Cooper and Mello[2])

When entering in a contract like this it is presumed that both parties obtain an economic benefit, and this will be a result of the principle of comparative advantage. The gain in this contract is simple, but both parties have a risk associated with their positions in the contract. Lets think of a firm that has a swap contract at a fixed rate interest. For a shot-term position a rise in the market rates is a gain, but a decline is a loss for the firm (Bicksler and Chen[1]). We explain this contract because is the most easiest contract to explain, of course, there is others, the main characteristic of a swap is there is a swap of an underling, and it can be a bond, a stock, a commodity, wherever the parties want, and it is traded with two parties, a buyer and a seller.

1.1.2 Description of a CDS

A CDS contract is a swap agreement between too counterparties where the main objective for the buyer is to protect a possible loss in the event of a default by a particular company. This loss (known as deliverable obligations that can be bonds or loans) were issued by the company known as reference entity. This is a negotiated over-the-counter contract and provides some insurance against the risk of the reference entity (O'Kane[9]). For such protection, the buyer needs to pay a stream of payments known as premium leg until a credit event or the maturity of the contract. If there is a credit event the protection seller will compensate the buyer; this compensation is known as protection leg. The premium leg is calculated from a default swap spread on the face value of the obligation.

Figure 1.1: Default Swap Premium Leg

The frequency of these payments is negotiated at the time of the contract (usually it is quarterly). The payment of the protection leg is the difference between the par and the price of the cheapest to deliver asset on the face value of the protection but the protection seller has to compensate the buyer for the loss O'Kane[9].

This payment can be settled in cash or physically:

Physically: The protection seller pays in cash the equivalent of the face value of deliverable obligation that was delivered by the protection buyer. In a physically delivery the buyer has the particular option to choose what is more convenient for him; he choose the cheapest asset to deliver and receives his face value in cash.

Cash: The protection leg is the difference between the face value of the protection and the recovery price of a specified bond or loan of the reference entity paid in cash.

Figure 1.2: The Protection Leg with Cash Settlement

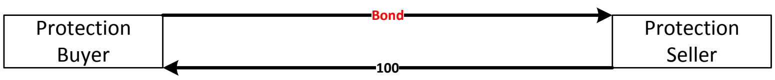

Figure 1.3: The Protection Leg with Physical Settlement

The details figured in the contract will be:

- Currency;
- Maturity of the contract;
- Reference entity;
- Notional;
- Default Swap Spread;
- Frequency;
- Payoff upon Default
- Credit Event.

1.1.3 Example of a CDS contract

An example announced in the O'Kane and Turnbull[11] paper will be presented:

Suppose that a protection buyer has a bond issued by some entity that may default in an uncertain time. So, to protect his position, he buys a 5-year protection on that bond with a face value of $10 million and a default swap spread of 300 bp. In the contract, the frequency of payments was defined to be quarterly, so the premium payments will be $\$10\,\text{million} \times 0.03 \times \frac{1}{4} = \$75\,\text{thousand}$ every three months.

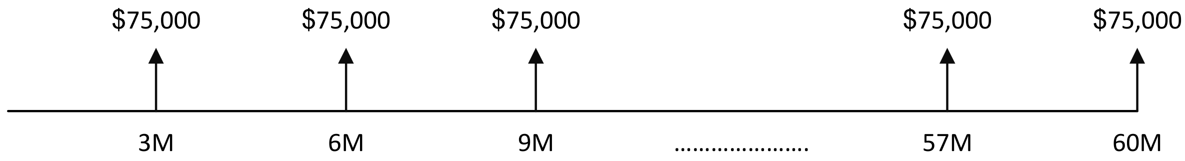

Figure 1.4: When no default occurs

Assume that the reference entity default after a short period of time, and that the cheapest to deliver (CTD) has a recovery rate of 45%.

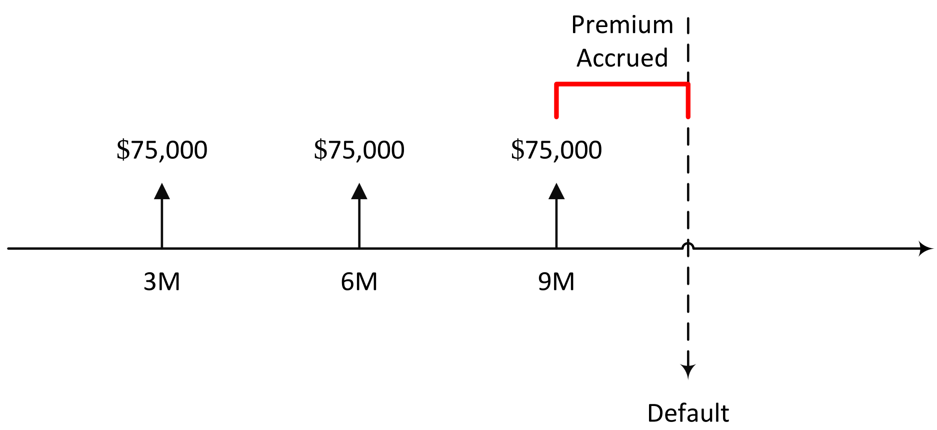

Figure 1.5: Default at time t with 45% recovery

At this time, the protection seller has to compensate the protection buyer with an amount of $\$10\,\text{million} \times (100\% - 45\%) = \$5,5\,\text{million}$ and if the credit event did not occurred in the time of the premium payment the protection buyer has to pay the accrued premium. Suppose that the credit event occurs after a month, then the accrued premium has to be $\$10\,\text{million} \times 0.03 \times \frac{1}{12} = \$18,780$.

2

Credit Modelling

Contents

2.1 Reduced Approach

There are two main approaches when modeling credit instruments: Structural Approach and Reduced-Form approach. In the Structural Approach modeling is based on the internal structure of the firm using historical data (Most models are usually an extension of Merton's 1974 firm-value model). For conducting such research, some information must be available about the balance sheet of the firm and it is not an easy way to model because such information is commonly published only about four times a year. The main goal of this approach is to create a link between the debt and equity of the firm and using such link to evaluate the probability of default of the firm. This type of modeling does not consider the spread trade in the market and shows some inconsistency with the risk neutral probability of the default used in the reduced-form approach.

For this work, a reduced-form approach will be used to price the credit derivative under analysis. This approach will use the default spread information, recovery rate, and interest rate risks to price and manage risk in a CDS contract.

When entering a CDS contract the buyer expects that his protection is secured in case of a credit event occurs, so the protection seller has to have no default. The same happens to the protection seller, he expects receiving a stream of payments to hedge his position in the contract - this type of risk is known as principal risk. So there are two probabilities to consider: the default probability of the buyer and the default probability of the seller. These probabilities will appear on the valuation of the premium leg and protection leg.

Using a reduced-form approach, a consistent and good model to price a CDS contract has to satisfy certain requirements:

1. Consider the risk of the default of the issuer;
2. Be able to calculate the risk of receiving the recovery rate;
3. The spread risk- the model needs to be able to see the change in the market credit spread even if no default occurred;
4. For a consistent model, it has to be arbitrage free;
5. It has to be a simple and flexible model to fit the term structure of prices of bonds, CDS and all kinds of credit derivatives.

The rating of the reference entity is not considered as a requirement since the credit spread will reflect the default risk of the issuer.

In a default model the main goal is to model the default time, τ, of a credit event. Assuming that default is an one-time event, since once occurred there will not be another one and once again assuming that every credit will eventually default, we can assume that τ is limited in the time interval $[0, \infty[$.

The model used is only interesting if default occurs before maturity time T, so $\tau < \mathrm{T}$ (O'Kane[10]).

2.2 Risk Neutral Pricing Framework

The idea of a risk-neutral framework is that every dealer which issued a certain derivative should hedge his position by trading the underling asset with no profit or costs. This type of hedging is considered to be dynamic since the market is continuously changing.

Since by this type of hedge no profit is received then the total value of the hedge positions plus the derivatives in a dealer portfolio should be zero. With this in mind, a risk-neutral portfolio is made.

As a result, the price of the derivative is the price that costs hedging it. To hedge it, the dealer usually resorts to a funding at a certain interest rate. This rate turns into the effective risk-free rate which the dealer mostly refers as Libor.

In this line of thinking, the value of the derivative contract is the expected value of the future payoff discounted to the valuation date, that is

$$\mathbf{V}(0) = \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{V}(T)}{\beta(0,T)}\right], \tag{2.1}$$

where the discount is made at the risk-free rate and the expectation under the risk-neutral measure, is given by

$$\beta(0,T) = \exp\left(\int_0^T \mathrm{r}(s)\,\mathrm{d}s\right). \tag{2.2}$$

Since the Libor is the value of $\$1$ at T time, the present value of Libor should be given as

$$\mathbf{Z}(0,T) = \mathbb{E}\left[\frac{1}{\beta(0,T)}\right] = \mathbb{E}\left[\exp\left(-\int_0^T \mathrm{r}(s)\,\mathrm{d}s\right)\right]. \tag{2.3}$$

It must be noted that there need to be some consideration when working with a risk-neutral framework. In a complete market, it is possible for the dealer to create a risk-free portfolio simple by finding hedging instruments that allows him to have a *risk free position.*

One essential requirement for a pricing model is to recreate all the market products and use them to value or hedge all the risks that this kind of transaction will implies. If this requirement is not fulfilled then the model can be considered as an arbitrage model and will not reflect if the derivative is a good investment.

No model can predict the changes in the market so the model that we are using to price the CDS is not an exception, it just can extract the information in the market and valuate the investment in the credit derivative.

There need to be some calibration to the prices of the market instruments since this is a requirement for a no-arbitrage model. A good model and a consistent model do not have the same meaning. The first depends on the requirements of the traders; the later is a model that can identify the existence of arbitrage in all products that have the same underlying risk but in the future, different payoffs.

Since the time of valuation is not always zero, in this work the time of pricing the credit derivative is *t*, so using equation (2.1) the value of a payoff at time *T* conditional on information to time t is,

$$\mathbf{V}(t) = \mathbb{E}\left[\frac{\mathbf{V}(T)}{\beta(0,T)}\middle|\mathcal{F}_t\right], \tag{2.4}$$

where $\mathcal{F}_t$ is the set of information up to and including t.

2.2.1 Modelling Credit

Zero Recovery Assumption

Lets think about a credit risky zero coupon bond, with a face value of $\$1$ with maturity at time T. We will assume that if default occurs there is no amount to be recovered. So if, default occurs before expiration date we receive nothing otherwise at T we receive $\$1$, as shown in figure (2.1).

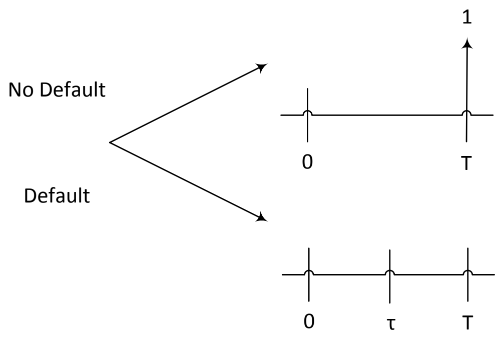

Figure 2.1: Zero Recovery Assumption

So the present value of this credit risky zero coupon bond will be the discounted expectation of receiving $\$1$ at maturity if no default occurs, and this is given by

$$\hat{\mathrm{Z}}(0,T) = \mathbb{E}\left[\exp\left(-\int_0^{\mathrm{T}} \mathrm{r}(s)\,\mathrm{d}s\right)\mathbf{1}_{\{\tau>T\}}\right], \tag{2.5}$$

where r(t) is the continuously compounded risk-free short rate, and the indicator function reflects the risk of default as follows:

$$\mathbf{1}_{\{\tau>T\}} = \begin{cases} 1 & \text{if } \tau > T \\ 0 & \text{if } \tau \leq T \end{cases}$$

Analyzing equation (2.5), and to price this risky zero coupon bond, we only need to know the probability that $\tau \leq T$. This is because any cash flow occurrence at a known time T. So we have a generic formula, it does not assume any dependence of τ. This expectation also allow for the possibility that there may be some co-dependence between the risk-free interest rate process and the default time.

Fixed Payment at Default Assumption

In Chapter 1, it was referred that the protection leg is a payment of an amount until the time of a credit event. This kind of amount has a risk of the protection buyer defaults, so it is not a riskless recovery. Consider now a simple credit risky structure that pays $\$1$ at the time of default τ if $\tau < \mathrm{T}$, and zero otherwise.

So, the price today, is given by discounting back the payment from the default time, that is

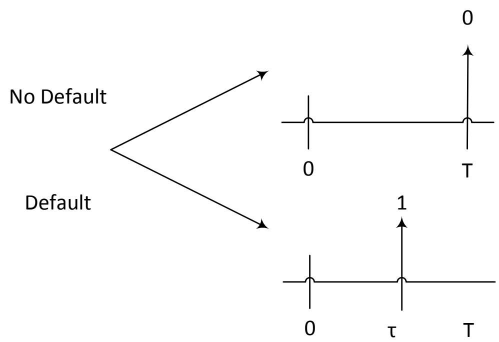

Figure 2.2: Fixed Payment Assumption Assumption

$$\hat{\mathrm{D}}(0,T) = \mathbb{E}\left[\exp\left(-\int_0^\tau \mathrm{r}(s)\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leq T\}}\right]. \tag{2.6}$$

The diference between this assumption and the previuos one is that now the time of the cash flow payment is unknown. If interest rates were zero we would only need to know if $\tau \leq \mathrm{T}$. But this does not happen in a real market, so we need to know not only the cumulative distribution $\mathrm{Pr}(t \leq \mathrm{T})$ but the full probability density of the default time distribution, i.e, $\mathrm{Pr}(\mathrm{t} \leq \tau \leq \mathrm{t} + \mathrm{dt})$.

Random Payment at Default Assumption

Finally, lets consider a credit risky structure which pays a random quantity $\Phi(\tau)$ at the time of default τ if $\tau \leq \mathrm{T}$, and zero otherwise.

So, as before, the price is given by

$$\hat{\mathrm{D}}(0,T) = \mathbb{E}\left[\exp\left(-\int_0^\tau \mathrm{r}(s)\,\mathrm{d}s\right)\Phi(\tau)\mathbf{1}_{\{\tau\leq T\}}\right]. \tag{2.7}$$

The only inconvenient of this assumption is the need to known if the size of the payment Φ may depend on the default time τ or even if it has any link with the interest rate process. So, once again, we need to know not only the cumulative distribution $\mathrm{Pr}(\tau \leq \mathrm{T})$ but the full probability density of the default time distribution, that is, $\mathrm{Pr}(\mathrm{t} < \tau \leq \mathrm{t} + \mathrm{dt})$.

This form will be used to value the protection leg if the quantity is random.

2.3 Modelling the Hazard Rate

For a better understanding of the Hazard Rate Model some definitions need to be considered first, using as reference the book of Gallager[(4)].

2.3.1 Poisson Process

In a simple way, a Poisson Process is a stochastic process mostly used for modeling arrival times. It is considered as being a continuous-time version of the Bernoulli Process.

In a Bernoulli Process the arrivals only occur at a positive integer whereas in a Poisson Process the arrivals may occur at an arbitrary time and for that reason the probability of an arrival at a particular time instant is 0.

Definition: An *arrival process* is defined as being an increasing sequence of random variables $S_1, S_2, \ldots, S_n$; these variables represent the time at which some event occurs.

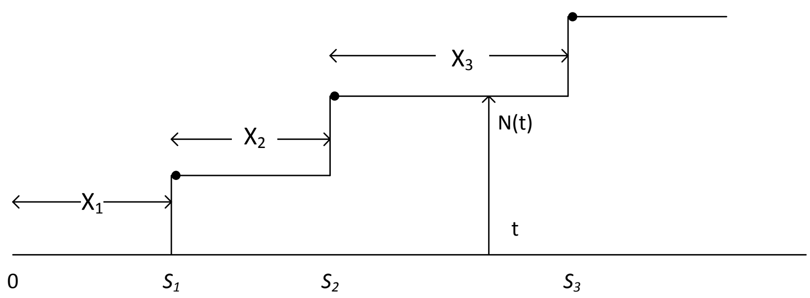

Figure 2.3: A function of an arrival process

We can describe an arrival process using stochastic processes. Consider the sequence $X_1, X_2, \ldots, X_n$ of the times between each arrival (we will call it interarrival times). It is easy to see that $X_1 = S_1, X_2 = S_2 - S_1$ and $X_i = S_i - S_{i-1}$ for $i > 1$ they will be positive random variables since they are defined using the arrival process. So we can describe the arrival process as

$$S_n = \sum_{i=1}^{n} X_i. \tag{2.8}$$

This is useful to specify the arrival process using the joint distribution of $X_1, X_2, \ldots, X_n$ for all $n > 1$.

An alternative specification will be using the counting process N(t), where for each $t > 0$, the random variable N(t) is the number of arrival up to and including t. So for each $t > 0$, N(t) is defined as the number of arrivals in the interval $]0, t]$. Without loss of generality, we can set N(0)=0 with probability 1.

As we can see in figure (2.3), the counting process $\{\mathrm{N}(t) : t > 0\}$ has the property that for every $\tau \geq t$, $\mathrm{N}(\tau) \geq \mathrm{N(t)}$. So $\mathrm{N}(\tau) - \mathrm{N(t)}$ is a non-negative random variable.

Definition: A *renewall process* is an arrival process for which the sequence of interarrival times is a sequence of positive random variables.

Definition: A *Poisson Process* is a renewall process in which the interarrival intervals have an exponential distribution function. For every $\lambda \in \mathrm{R}^+$ each X_i has the density $f_X(x) = \lambda \exp(-\lambda x)\,\mathrm{for}\, x \geq 0$ we will denote λ, as being the rate of the process.

A random variable is considered to be **Memoryless** if

$$\Pr(X > t + x | X > t) = \Pr(X > x)$$

With the memoryless property of exponentials random variables we can find the distribution of the first arrival in a Poisson Process after a given arbitrary time t>0.

Theorem: *For a Poisson Process of rate λ, and any given $t > 0$, the lenght of the interval from t until the first arrival after t is a non negative random variable Z with the distribution function* $1 - \exp(-\lambda z)\, for\, z \geq 0$

In other words, the idea is that Z, conditional on the time τ of the last arrival before t, is the remaining time until the next arrival

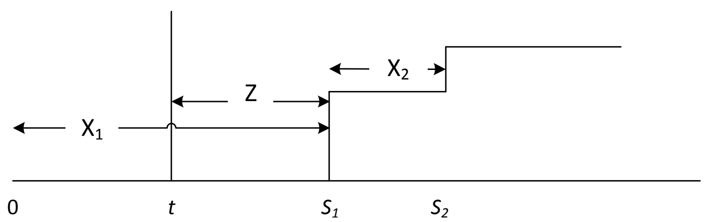

Figure 2.4: For arbitrary fixed $t > 0$, consider the event $N(t) = 0$. Conditional on this event, Z is the distance from t to S_1

It is important to know that the time until the first arrival after t is an exponentially distributed random variable with parameter λ, and all the others interarrival intervals, X_i are independent of this first arrival and of each other and all have the same exponential distribution

Definition: A *counting process* $\{\mathrm{N}(t) : t > 0\}$ has the stationary increment property if $N'(t) - N(t)$ has the same distribution function as $N(t' - t)$ for every $t' > t > 0$.

Assuming that $\mathrm{N'(t)} - \mathrm{N(t)} = \tilde{\mathrm{N}}(\mathrm{t}, \mathrm{t'})$ as the number of arrivals in the interval $]t, t']$ for any given $t' \geq t$, this definition states that the distribution of the number of arrivals in an interval depends on the size of the interval but not on its starting point.

Definition: A *counting process* $\{\mathrm{N}(t) : t > 0\}$ has the independent increment property if for every

$k \in \mathbb{N}$ and every *k-tuple of times* $0 < t_1 < t_2 < \ldots < t_k$. the k-tuple of random variables $N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t_k)$ are independent.

Theorem: *Poisson Processes* have both the stationay increment and independent increment properties.

We already saw that for a Poisson Process $S_n = \sum_{i=1}^{n} X_i$ and each random variable X_i has the density function as $f_X(x) = \lambda \exp(-\lambda x) \, \text{for} \, x \geq 0$. The density of the sum of two independent variables can be found by parcel their densities.

Let us start for the first two variables in the Poisson Process. So for $n = 1$, we have $S_1 = X_1$, for $n = 2$, $S_2 = X_1 + X_2$, and $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$, doing $X_2 = S_2 - X_1$, the density function is then given by

$$f_{X_1,S_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(s_2 - s_1).$$

For an independent and identically distributed (IID) exponential random variables X_1, X_2 the joint density function is then

$$\begin{aligned} f_{X_1,S_2}(x_1, x_2) &= \lambda \exp(-\lambda x_1) \times \lambda \exp(-\lambda(s_2 - x_1)) \\ &= \lambda^2 \exp(-\lambda s_2) \text{ for } 0 \leq x_1 \leq s_2. \end{aligned} \tag{2.9}$$

And for the sum of an arbitrary n of IID exponential random variables such as S_n, substituing $S_n - X_1 - \ldots - X_{n-1}$ for X_n we have

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= f_{X_1,\ldots,X_{n-1},S_n}(x_1, \ldots, x_{n-1}, s_n) \\ &= \lambda^n \exp(-\lambda s_n) \end{aligned} \tag{2.10}$$

Replacing the interarrival intervals $X_1, \ldots, X_n$ by what we want such as $S_1, \ldots, S_n$ where $S_1 = X_1$ and $S_i = S_i + S_{i-1}$ for $2 \leq i \leq n-1$, the specification of a Poisson Process will be

$$f_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = \lambda^n \exp(-\lambda s_n) \text{ for } 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n. \tag{2.11}$$

Integrating this over s_1 then s_2 and so on we get

$$f_{S_n} = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!} \text{ for } 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n. \tag{2.12}$$

In the following section it will be necessary to know the probability mass function of a Poisson counting process $\{\mathrm{N}(t) : t > 0\}$, since we have to work with some probabilities of default that uses these counting processes. So it is important to enunciate the next theorem.

Theorem: For a Poisson Process of rate λ, and for any $t > 0$, the probability mass function (pmf) of the number of arrivals in $]0, t]$ (N(t)) is given by the Poisson pmf

$$p_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}. \tag{2.13}$$

Non-Homogeneous Poisson Process

In this explanation of a Poisson Process, the arrival rate λ is assumed to be constant, but this is a very simple case and to generalize the process it is useful to consider λ as a function of time.

So a non-homogeneous Poisson Process is a counting process $\{\mathrm{N}(t) : t > 0\}$ which has the independent increment property with $\lambda(t)$ as a function of time and it has the pmf shown in the theorem. The only difference from the previous definition is that a non-homogeneous Poisson Process does not have the stationary increment property.

As in an homogeneous Poisson Process, the next theorem follows:

Theorem: For a non-homogeneous Poisson Process with right-continuous arrival rate $\lambda(t)$ bounded away from zero, the distribution of the number of arrivals in $]t,\tau]$, $\tilde{N}(t,\tau)$ satisfies,

$$\begin{aligned}\Pr\{\tilde{N}(t,\tau) = n\} =& \frac{[\tilde{m}(t,\tau)]^n}{n!} \exp\left[\tilde{m}(t,\tau)\right] \; where \, \tilde{m}(t,\tau) \\ =& \int_t^T \lambda(u) \, \mathrm{d}u.\end{aligned}$$

When working with this kind of processes it is obviously that the most interesting thing is to condicionate them to some interval event so the following theorem will be essential when calculating the probabilities in the next section.

Theorem: Let $f_{(S_1,\ldots,S_n|N(t)=n)}(s_1,\ldots,s_n|n)$ be the joint density of $S^{(n)} = (S_1,\ldots,S_n)$ conditional on $N(t) = n$. This density is constant over the region $0 < s_1 < \ldots < s_n < t$ and has the value $f_{S^{(n)}|N(t)=n}(s^{(n)}|n) = \dfrac{n!}{t^n}$

The framework used to model the derivative was initially established in Jarrow and Turnbull[(7)] and later in David[(3)]. In the next section an introduction of this framework will be explained. This will be essential to calibrate the survival probability used to valuate the credit derivative.

2.3.2 Hazard Rate Model

The idea is to model default as the first arrival time τ of a Poisson Process. As it was described, a non-homogeneous Poisson Process N with (non-negative) intensity function $\mathrm{l}(.)$ satisfies

$$\mathrm{P}(N_t - N_s = k) = \frac{\left(\int_s^t \mathrm{l}(u)\,\mathrm{d}u\right)^k}{k!} \exp\left(-\int_s^t \mathrm{l}(u)\,\mathrm{d}u\right), \mathrm{with}\, k \in \mathbb{N}_0. \tag{2.14}$$

Assuming $\mathrm{N}_0 = 0$, then

$$\mathrm{P}(N_t = 0) = \exp\left(-\int_0^t \mathrm{l}(u)\,\mathrm{d}u\right). \tag{2.15}$$

A way of simulating the first jump τ of N is to let E_1 to be a unit exponential random variable and define the first arrival time τ as

$$\tau = \inf\left\{t : \int_0^t \mathrm{l}(u)\,\mathrm{d}u \geq \mathrm{E}_1\right\}. \tag{2.16}$$

A Cox Process is a generalization of a Poisson Process, since the first is known as a doubly stochastic Poisson Process and is mostly used to valuate financial instruments in which credit risk is an important factor (David[(3)]).

Since it is a doubly stochastic Poisson Process, the intensity function can be random but in such a way that if conditional on a particular realization $l(\cdot, \omega)$ of the intensity, the jump process becomes an inhomogeneus Poisson Process with intensity $l(s, \omega)$. This will be known as $\lambda(X_s) = l(s, \omega)$ where X_s is an $\mathbb{R}^d$-valued stochastic process and $\lambda : \mathbb{R}^d \to [0, \infty[$ is a non-negative continuous function that we will call hazard rate.

Since a firm survives up to time t, and given the history of X up to time t, the probability of defaulting within the next small time interval dt is equal to $\lambda(X_t)dt + dt$.

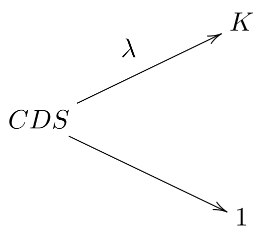

Figure 2.5: One Period

For one period there is only two possibilities until maturity: Default or No Default.

- If there is a default the payoff will be less than the face value of the bond.
- If there is no default the payoff is the face value of the bond.

For simplifying the framework, the payoff in the event of default will be an exogenous constant.

This payoff in case of default, is denoted by K and is assumed as being the same for all instruments in a given credit risk class since different classes of debt in the same firm can have different recovery rates in different times (Jarrow and Turnbull[(7)]). Then K can be greater for those classes and the consequently payoff greater for different times. This discrete-time binomial process was selected to approximate a continuous-time Poisson Process.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space large enough to support an $\mathbb{R}^d$-valued stochastic process $X = \{\mathrm{X}_t : 0 \leq t \leq T\}$ which is right-continuous with left limits and a unit exponential random variable E_1 which is independent of X, and $\lambda : \mathbb{R}^d \to \mathbb{R}$ a non-negative and continuous function. Using equation (2.16), the default time τ can be defined as:

$$\tau = \inf\left\{ t : \int_0^t \lambda(X_s)\,\mathrm{d}s \geq \mathrm{E}_1 \right\}. \tag{2.17}$$

As it was explained in David[3], when $\lambda(X_s)$ is large the integrated hazard grows faster and reaches the level of the independent exponential variable faster and thus the probability that τ is small becomes higher.

By definition, the two relationships will follow:

$$\Pr(\tau > t|\,(\mathrm{X}_s)_{0\leq s\leq t}) = \exp\left(-\int_0^t \lambda(X_s)\,\mathrm{d}s\right) \text{ with } t \in [0,T]\,, \tag{2.18}$$

and

$$\Pr(\tau > t) = \mathrm{E}\left[\exp\left(-\int_0^t \lambda(X_s)\,\mathrm{d}s\right)\right] \text{ with } t \in [0,T]\,. \tag{2.19}$$

The model for the default-free term structure of interest rates uses a spot rate process and some expectations, such like the expectation of the money market account and the expectation of the equivalent martingale measure

$$\beta(0,t) = \exp\left(\int_0^t \mathrm{r}(s)\,\mathrm{d}s\right) \tag{2.20}$$

and

$$\mathrm{p}(t,T) = \mathbb{E}\left(\frac{\beta(t)}{\beta(T)}|\mathcal{F}_t\right). \tag{2.21}$$

Following the framework described above, the information at t is given by, $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, where

$$\mathcal{H}_t = \left\{\mathbf{1}_{\{\tau\leq\, s\}} : 0 \leq\; s \leq\; t\right\}$$

holds the information of whether there has been a default at time t and

$$\mathcal{G}_t = \{\mathrm{X}_s : 0 \leq\; s \leq\; t\}$$

holds the information of the value of the spot rate and hazard rate of default.

To price the CDS it will be needed the promised payments and its payment in the event of default. In Lando (1998) this approach is based on the following basic building blocks:

- $X\mathbf{1}_{\{\tau>T\}}$: A payment $X \in \mathcal{G}_T$ at a fixed date which occurs if there has been no default before time T.
- $Y_s\mathbf{1}_{\{\tau>s\}}$: A stream of payments at a rate specified by the $\mathcal{G}_t$-adapted process Y which stops when default occurs (Protection Leg).
- $\mathcal{Z}_t$: A recovery payment at the time of default of the form $\mathcal{Z}_t$ where Z is a $\mathcal{G}_t$-adapted stochastic process and $\mathcal{Z}_t = \mathcal{Z}_{\tau(\omega)}\,(\omega)$

With this information, the computing of the probabilities used in this framework will be presented in the next section.

2.3.3 Survival Probability

To simplify the model it will be used a deterministic hazard rate. Looking to figure (2.5) the probability of default is given by:

$$Pr\,(\tau < t + dt|\tau > t) = \lambda\,(t)\,dt. \tag{2.22}$$

With this definition, the survival probability from time 0 to time T is obtained as

$$\Pr\left(\tau > T\right) = (1 - \lambda(dt)dt)(1 - \lambda(2dt)dt)\dots(1 - \lambda(dT)dt). \tag{2.23}$$

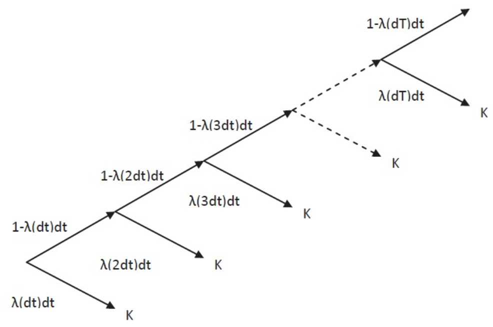

Figure 2.6: The survival probability from time 0 to time T

Integrating over time t, and setting the limit $dt \to 0$,

$$\Pr(\tau > T) = \exp\left(-\int_0^T \lambda(t)\,\mathrm{d}t\right). \tag{2.24}$$

Differentiating the cumulative distribution function the probability density of default time is given by,

$$\begin{aligned}\Pr(T < t \leq T + dT) &= -\frac{\partial \Pr(\tau > T)}{\partial T} dT \\ &= -\frac{\partial}{\partial T}\left[\exp\left(-\int_0^t \lambda(t)\,\mathrm{d}t\right)\right] dT \\ &= -\exp\left(-\int_0^t \lambda(t)\,\mathrm{d}t\right)\frac{\partial}{\partial T}\left(-\int_0^t \lambda(t)\,\mathrm{d}t\right) dT. \end{aligned} \tag{2.25}$$

2.4 The Credit Triangle

This concept is essential to know why the gain or loss from a CDS position cannot be the difference between the current market quoted price plus the coupons received and the purchase price.

Consider a contract linked to an issuer with a deterministic hazard rate, maturity time T and the contract pays $(1 - \mathrm{R})$ at the time of a credit event if default occurs before maturity.

For such protection the buyer pays a stream of payments on a spread S until default or maturity whichever occurs first. With this information, a valuation of the premium and protection leg needs to be made.

For the premium leg, between time t and $t + \mathrm{d}t$ the protection buyer pays $\mathrm{S}\mathrm{d}t$ if the credit has not defaulted. Discounting this payment with the Libor factor and integrating over the lifetime of the contract gives

$$\mathrm{Expected\,Present} - \mathrm{Value\,of\,a\,Tyear\,Premium\,Leg\,at\,a\,S\,bp} = S \times \int_0^T \mathrm{Z}(0,t) \times \mathrm{Q}(0,t)\,\mathrm{d}t. \qquad (2.26)$$

For the protection leg, a payment of $(1-\mathrm{R})$ is made if default occurs

$$\mathrm{Protection\,Leg} = (1-\mathrm{R}) \times \int_0^T \mathrm{Z}(0,t) \times \lambda(t)\mathrm{Q}(0,t)\,\mathrm{d}t. \qquad (2.27)$$

Assuming a deterministic hazard-rate this equation becomes

$$\mathrm{Protection\,Leg} = \lambda(1-\mathrm{R}) \times \int_0^T \mathrm{Z}(0,t)\mathrm{Q}(0,t)\,\mathrm{d}t. \qquad (2.28)$$

Since in the beginning of the contract the value of the contract is zero, then the premium leg and the protection leg are equal, so using (2.26) and (2.28) the value of the spread is given by

$$\mathrm{S} = \lambda(1-\mathrm{R}). \qquad (2.29)$$

This relationship is called ***the credit triangle*** since it is a function of three variables and knowing two of them is sufficient to calculate the remain.

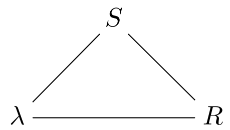

Figure 2.7: Credit Triangle.

Analyzing equation (2.29), it states that the required continuously paid spread compensation for taking on a credit loss of $(1-\mathrm{R})$ equals the hazard rate times $(1-\mathrm{R})$.

3

Implementation of the Model

Contents

3.1 The Mark-To-Market Value

To introduce the model offered by O'Kane and Turnbull[11] let us first describe a simple example of a CDS contract. Suppose an investor buys a 5-year protection on a company with a default swap spread of 60 bp. After 1 year, the same investor wants to value his position on the contract. At this point, the 4-year credit default swap spread quoted in the market is 170 bp. So the default swap spread has risen over the past year and the buyer is then paying 60 bp for a protection that the market values now in 170 bp. The investor wants to know his current position value.

The Mark-to-Market (MTM) value is simply given by the difference between the current market value of the remaining 4-year protection and the expected present value of the 4-year premium leg at 60bp, i.e.

MTM $=$ Current Market Value of Remaining 4-year Protection $-$ Expected Present Value of 4-year Premium Leg at 60bp.

Since the value of a new credit default swap is zero, we have

Current Market Value of Remaining 4-year Protection $=$ Expected Present Value of 4-year Premium Leg at 170bp.

Therefore, the market-to-market value to the protection buyer is given by

MTM $=$ Expected Present Value of 4-year Premium Leg at 170bp $-$ Expected Present Value of 4-year Premium Leg at 60bp.

Defining the expected present value of 1bp paid on the premium leg until default or maturity as Risky PV01, we can rewrite the above expression as

$$\begin{aligned}\mathrm{MTM} &= \mathrm{Risky\ PV01} \times 170\mathrm{bp} - \mathrm{Risky\ PV01} \times 60\mathrm{bp}\\ &= 110\mathrm{bp} \times \mathrm{Risky\ PV01}.\end{aligned}$$

Setting this as an example and generalizing it we get the mark-to-market value of the position initially traded:

$$\mathrm{MTM} = \pm\left[\mathrm{S}(\mathrm{t_V}, \mathrm{t_N}) - \mathrm{S}(\mathrm{t_0}, \mathrm{t_N})\right] \times \mathrm{RPV01}(t_V, t_N), \tag{3.1}$$

where

t_N**-** is the maturity of the initially traded contract;

t_V**-** is the time of the valuation position;

t_0**-** is the initial time of the contract;

$S(t_0, t_N)$**-** The initially contractual spread;

$S(t_V, t_N)$**-** The spread at the valuation date;

$RPV01(t_V, t_N)$**-** The present value at time t_V of a 1bp premium stream which terminates at maturity t_N or default.

If the investor is in a **long protection position** the $+$ sign is used, whereas the - sign stands for a **short protection position**.

Equation (3.1) highlights that the only remaining ingredient that needs to be calculated is the RPV01. To accomplish this task we need a model that captures the risk of every payment in the premium leg and the final payment on maturity or at the time of a credit event.

In the next section, the valuation of both the Premium Leg and the Protection Leg will be made using the insights explained in Chapter 2. For both valuations we will need to determinate the survival probability of both positions (buyer and seller). For the buyer it is obvious that there may be a default event and, therefore, the premium payments are risky. In the case of the seller, the risk is implied when there is a large amount to pay, or in case of a credit event, or maturity, but both will be explained in detail in the next section.

3.2 Modelling a CDS contract

In this section we will begin to valuate a CDS contract based on the reduced-form approach introduced in Chapter 2. Following the standard assumption, we will assume that the hazard rate process, interest rates, and recovery rates are independent. We will implement the so-called standard model of O'Kane and Turnbull[11]. An exposition of such model is also available in the book of O'Kane[10].

The following notation will be used throughout this chapter:

t**-** is the effective date;

$\mathrm{Z(t, T)}$**-** is the Libor discount curve;

$\mathrm{Q(t, T)}$**-** is the survival probability up to T;

$\mathrm{t_n}$**-** for $n = 1, \cdots, N$ are the premium payment dates. We set $t_0 = t$;

T**-** corresponds to when the protection end;

$\mathrm{S_0}$**-** represents S(0,T), i.e, the fixed contractual spread of a contract traded at time 0 which matures at time T;

$\Delta(\mathrm{t_{n-1}, t_n})$ - is the day count fraction between dates t_{n-1} and t_n in the appropriate day count convention, typically Actual/360;

R**-** is the expected recovery rate as a percentage of par.

3.2.1 Valuing The Premium Leg

In the beginning of this thesis, it was explained how a CDS contract works and what is the premium leg. As a remainder, the premium leg is defined as the set of payments that the buyer has to do to the protection seller to insure his protection. So this payments are made until default or maturity, whichever occurs first. In case of default happens in a time that does not coincide with the payment date of the premium payment, then the buyer must pay the premium which has accrued from the previous premium payment date.

For the valuation, the case when there are no accrued payments is easier to introduce. Hence, for now, we will present the valuation without accrued payments or assuming that the credit event has occurred on the premium payment date.

In Chapter 2 we have introduced the zero recovery risky zero coupon bond assumption, i.e, a contract that pays $\$1$ if there is no default before T. So the premium payments follow this type of recovery. As it was shown, the present value of $\$1$ which is to be paid at time t_n, but is canceled if default occurs before maturity with zero recovery is given by

$$\hat{\mathrm{Z}}(\mathrm{t},\mathrm{t_n}) = \mathbb{E}\left[\exp\left(-\int_t^{t_n} \mathrm{r}(s)\,\mathrm{d}s\right)\mathbf{1}_{\tau>t_n}|\mathcal{G}_t\right], \tag{3.2}$$

where r(t) is the continuously compounded risk-free rate.

As stated before, the short interest rate process and the default time are assumed to be independent, so the expected present value becomes

$$\begin{aligned}\hat{\mathrm{Z}}(\mathrm{t},\mathrm{t_n}) &= \mathbb{E}\left[\exp\left(-\int_t^{t_n} \mathrm{r}(s)\,\mathrm{d}s\right)|\mathcal{G}_t\right] \times \mathbb{E}\left[\mathbf{1}_{\tau>t_n}|\mathcal{G}_t\right] \\ &= \mathrm{Z}(t,t_n) \times \mathrm{Q}(\tau > t_n|\mathcal{F}_t) \\ &= \mathrm{Z}(t,t_n) \times \mathrm{Q}(t,t_n). \end{aligned} \tag{3.3}$$

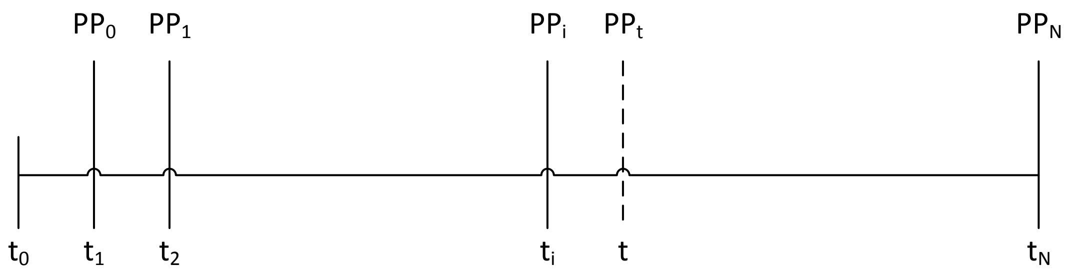

Figure 3.1: Premium Payments of the Premium Leg.

Since the premium leg is the sum of all premium payments made on the spread S_0, the present value becomes

$$S_0 \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Q(t, t_n) Z(t, t_n). \tag{3.4}$$

This equation can be resumed as being the sum of each premium payment weighting each by the probability of surviving to the payment date and then discounted the payment back to today at the risk free rate.

Now we will see the effect of premium accrued in the premium leg for the valuation of the CDS. The premium accrued is the payment that needs to be made if default time is different than the premium payment date. This is the amount of premium which has accrued from the previous payment date to the default time. This amount is then determined by the date of the credit event if a credit event occurs. This is a contingent payment that was discussed in subsection 2.2.1, where we showed that the price today of $\$1$ paid at default which occurs in the interval $[s, s+ds]$ is given by

$$Z(t, s)(-dQ(t, s)). \tag{3.5}$$

So the expected present value of the premium accrued due to default in the interval s to $s+ds$ in the nth premium period is then given by

$$S_0 \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)). \tag{3.6}$$

Since default can happen any time, we cannot work in discrete time, so the value of the premium accrued during this coupon period is

$$S_0 \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)). \tag{3.7}$$

Similarly to what we have done before, we need to sum over all of the premium payments so we can calculate the expected present value of the premium accrued. As a result, we have the following equation

$$S_0 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)). \tag{3.8}$$

Hence, we are able to write the present value of the premium leg as:

$$\begin{aligned} \text{PV Premium Leg} &= S_0 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)) + S_0 \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Q(t, t_n) Z(t, t_n) \\ &= S_0 \times \left(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)) + \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Q(t, t_n) Z(t, t_n) \right) \\ &= S_0 \times \text{RPV } 01(t, T), \end{aligned} \tag{3.9}$$

where we will denote RPV01(t,T) as the risky PV 01, i.e. the expected present value of 1bp paid on the premium leg until default or maturity. Now, we need to solve the integration in the second term. The problem of this integral is that all the functions are not deterministic and, therefore, it becomes difficult to integrate the product. But we can approximate this integral by simply thinking that if there is a default between two consecutive times on average it will occur halfway this period and so the accrued premium at default will be $S_0\Delta(t_{n-1}, t_{-n})/2$.

The probability of defaulting during the nth premium payment period is given by $Q(t, t_{n-1}) - Q(t, t_n)$. Discounting to today using the discount factor to the end of the period, the result of the approximation is

$$\approx \frac{1}{2}\Delta(t_{n-1}, t_n)\mathrm{Z}(t, t_n)\left[\mathrm{Q}(t, t_{n-1}) - \mathrm{Q}(t, t_n)\right]. \tag{3.10}$$

We notice that to be consistent with the approximation, we should use a discount factor to time $(t_{n-1}+t_n)/2$, but this would imply calculating another discount factor; with this approximation the final form of PV01 becomes simpler.

This approximation has an error in the risky premium leg PV01 of $\mathcal{O}(10^{-5})$ and for the present value of the premium leg we need to multiply it by the spread that is of order of $\mathcal{O}(10^{-2})$ and so the final Present Value error on a CDS contract with face value of $\$1$ will be about $\mathcal{O}(10^{-7})$ which is a very reasonably level.

Finally, next equation represents the present value of the premium leg risky PV01 which incorporates the risky premium payments and the payment of coupon accrued at default:

$$\begin{aligned}\mathrm{RPV01}(t, T) = &\sum_{n=1}^{N}\Delta(t_{n-1}, t_n)\mathrm{Q}(t, t_n)\mathrm{Z}(t, t_n)\\ &+\frac{1}{2}\sum_{n=1}^{N}\Delta(t_{n-1}, t_n)\mathrm{Z}(t, t_n)\left[\mathrm{Q}(t, t_{n-1}) - \mathrm{Q}(t, t_n)\right].\end{aligned} \tag{3.11}$$

We can approximate the spread adjustment implied by the specification of coupon accrued. Defining f as the annual frequency of the premium payment, the probability of the issuer defaulting in the ith premium payment period of length $1/f$ years conditional on surviving to the start of the period is given by λ_i/f. Using the same argument as before, if default happens in a coupon period, on average the payment of the premium accrued will be $S/2f$. So we multiply each potential payment by the probability of surviving to the start of each coupon period and then discounting back to today. The result is then given by,

$$(S/2f)\times(1/f)\sum_{i=1}^{N}\mathrm{Q}_i\mathrm{Z}_i\lambda_i. \tag{3.12}$$

Dividing by the Risky PV01, which is given by $(1/f)\sum_{i=1}^{N}\mathrm{Q}_i\mathrm{Z}_i$, we get this into spread terms. Assuming that the CDS curve is flat, to have $\lambda_i = \lambda$, then the spread impact is $S\lambda/(2f)$. Using the relationship demonstraded in section 2.4 we can write

$$\lambda \approx S(1-R). \tag{3.13}$$

Finally, we have the effect as being:

$$\text{S(Without accrued at default)} - \text{S(With accrued at default)} = \frac{\mathrm{S}^2}{2(1-\mathrm{R})f}. \tag{3.14}$$

Valuing between Premium Payment Dates

What if we want to value our CDS between premium payments? If this is the case, then equation (3.9) does not take into account the accrued premium today, and so we need to improve our formula so this first cash flow after today can be considered.

Setting t_n as the time of this cash flow we can write the premium present value as

$$\begin{aligned}\text{Premium PV} = \mathrm{S}_0 &\int_t^{t_{n^*}} \Delta(t_{n^*-1}, s)\mathrm{Z}(t,s)(-d\mathrm{Q}(t,s)) \\ &+ \mathrm{S}_0 \sum_{n=n^*+1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s)\mathrm{Z}(t,s)(-d\mathrm{Q}(t,s)) \\ &+ \mathrm{S}_0 \sum_{n=n^*}^{N} \Delta(t_{n-1}, t_n)\mathrm{Q}(t,t_n)\mathrm{Z}(t,t_n). \end{aligned} \tag{3.15}$$

The first term in this equation takes into account the fact that even if the credit event occurs today the buyer receives the fraction of the next coupon which has accrued since the last coupon date. We must note that the accrual factor in the integral starts in t_{n-1}, whereas the integral starts from today.

We notice that the payment corresponding to this small amount of time is actually risk free. We will now calculate the integrals that are in equation (3.15). It is easy to see that:

$$\begin{aligned}\mathrm{S}_0 \sum_{n=n^*}^{N} \Delta(t_{n-1}, t_n)\mathrm{Q}(t,t_n)\mathrm{Z}(t,t_n) = {} & \mathrm{S}_0 \Delta(t_{n^*-1}, t_{n^*})\mathrm{Q}(t,t_{n^*})\mathrm{Z}(t,t_{n^*}) \\ &+ \mathrm{S}_0 \sum_{n=n^*+1}^{N} \Delta(t_{n-1}, t_n)\mathrm{Q}(t,t_n)\mathrm{Z}(t,t_n). \end{aligned} \tag{3.16}$$

For the second term

$$\mathrm{S}_0 \sum_{n=n^*+1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s)\mathrm{Z}(t,s)(-d\mathrm{Q}(t,s)), \tag{3.17}$$

we will use the approximation that was explained earlier, which implies that if default occurs during the coupon period, on average it will occur halfway, and so, half of the accrued coupon will be paid

$$\approx \frac{\mathrm{S}_0}{2} \sum_{n=n^*+1}^{N} \Delta(t_{n-1}, t_n)\mathrm{Z}(t,t_n)\left[\mathrm{Q}(t,t_{n-1}) - \mathrm{Q}(t,t_n)\right]. \tag{3.18}$$

And finally, for the last term (i.e. the first term on the right-hand size of equation (3.15)), we have

$$\begin{aligned}\mathrm{S}_0 \int_t^{t_{n^*}} \Delta(t_{n^*-1}, s)\mathrm{Z}(t,s)(-d\mathrm{Q}(t,s)) = {} & \mathrm{S}_0 \Delta(t_{n^*-1}, t)\mathrm{Z}(t,t_{n^*})\left[1 - \mathrm{Q}(t,t_{n^*})\right] \\ &+ \frac{\mathrm{S}_0}{2} \Delta(t, t_{n^*})\mathrm{Z}(t,t_{n^*})\left[1 - \mathrm{Q}(t,t_{n^*})\right]. \end{aligned} \tag{3.19}$$

The buyer has to receive the fraction between the previous coupon and the valuation date, with probability of defaulting in the next period, so this probability is given by $Q(t,t) - Q(t,t_{n^*}) = 1 -$

$Q(t, t_{n^*})$, discounting to t using the discount factor to the end of the period we have the first term of equation (3.19).

The second follows the argument described earlier, which assumes that if default occurs during the coupon period, on average it will occur halfway and so half of the accrued coupon will be paid.

In summary, joining equations, we can write the Premium PV of the premium leg as

$$\begin{aligned}\text{Premium PV} &= \mathrm{S}_0\Delta(t_{n^*-1}, t)\mathrm{Z}(t, t_{n^*})\left[1 - \mathrm{Q}(t, {t_n}^*)\right] \\ &+ \frac{\mathrm{S}_0}{2}\Delta(t, t_{n^*})\mathrm{Z}(t, t_{n^*})\left[1 - \mathrm{Q}(t, t_{n^*})\right] \\ &+ \mathrm{S}_0\Delta(t_{n^*-1}, t_{n^*})\mathrm{Q}(t, t_{n^*})\mathrm{Z}(t, t_{n^*}) \\ &+ \frac{\mathrm{S}_0}{2}\sum_{n=n^*+1}^{N}\Delta(t_{n-1}, t_n)\mathrm{Z}(t, t_n)\left[\mathrm{Q}(t, t_{n-1}) + \mathrm{Q}(t, t_n)\right] \\ &= \mathrm{S}_0 \times \mathrm{RPV01}(t, T), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\mathrm{RPV01}(t, T) &= \Delta(t_{n^*-1}, t)\mathrm{Z}(t, t_{n^*})\left[1 - \mathrm{Q}(t, {t_n}^*)\right] \\ &+ \frac{1}{2}\Delta(t, t_{n^*})\mathrm{Z}(t, t_{n^*})\left[1 - \mathrm{Q}(t, t_{n^*})\right] \\ &+ \Delta(t_{n^*-1}, t_{n^*})\mathrm{Q}(t, t_{n^*})\mathrm{Z}(t, t_{n^*}) \\ &+ \frac{1}{2}\sum_{n=n^*+1}^{N}\Delta(t_{n-1}, t_n)\mathrm{Z}(t, t_n)\left[\mathrm{Q}(t, t_{n-1}) + \mathrm{Q}(t, t_n)\right]. \end{aligned} \tag{3.21}$$

3.2.2 Valuing The Protection Leg

As we have explained before, the protection leg is the payment that the protection seller has to pay to the buyer after a credit event occurs. This is a contingent payment of $100\% - R$ (where R is the expected recovery rate) of the face value of the protection. It is an uncertain payment, since before that, it has to occur a credit event and the protection seller need to have the amount in question. The modeling of such payment has been discussed in Section 2.2, where we have shown that the price of a security which pays an uncertain quantity $\Phi(\tau)$ at the time of default $\tau, \text{ if } \tau \leq T$, is given by

$$\hat{\mathrm{D}}(t, T) = \mathbb{E}\left[\exp\left(-\int_t^\tau \mathrm{r}(s)\,\mathrm{d}s\right)\Phi(\tau)\mathbf{1}_{\{\tau\leq T\}}|\mathcal{G}_t\right]. \tag{3.22}$$

We have assumed that the expected present value of the protection payment value of the protection payment $1 - R$ is independent of interest rates and the default time, and the default time is also independent of interest rates, so we can rewrite the above equation as

$$\hat{\mathrm{D}}(t, T) = \mathbb{E}\left[\exp\left(-\int_t^\tau \mathrm{r}(s)\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leq T\}}|\mathcal{G}_t\right]\mathbb{E}\left[\Phi(\tau)\right],$$

and, therefore,

$$\text{Protection PV}(t, T) = (1 - \mathrm{R})\int_t^T \mathrm{Z}(t, s)(-d\mathrm{Q}(t, s)). \tag{3.23}$$

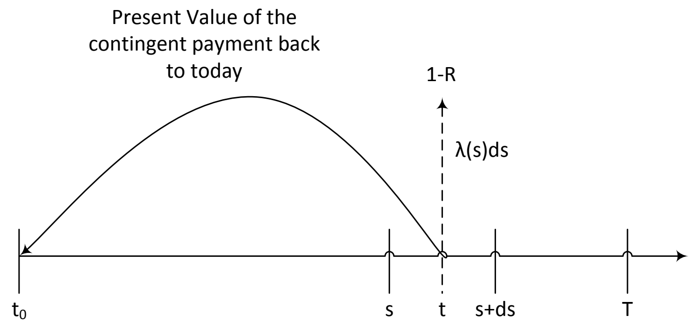

Figure 3.2: Valuing Protection Leg. Source O'Kane[10]

As seen in O'Kane[10] this integral over s can be performed by discretizing the time between t and T into $K = \mathrm{int}(M \times (\mathrm{T} - t) + 0.5)$ equal intervals, where M is the number of integration steps per year. As we increase M, the accuracy of this integral will improve.

We will now introduce a way to improve this accuracy. Since $\mathrm{Z}(t, T)$ is a monotonically decreasing function of T, we can bound the protection leg. The lower bound is then given by:

$$\mathcal{L} = (1 - \mathrm{R}) \sum_{k=1}^{K} \mathrm{Z}(t, t_k) \left[\mathrm{Q}(t, t_{k-1}) - \mathrm{Q}(t, t_k) \right], \tag{3.24}$$

and the upper bound by

$$\mathcal{U} = (1 - \mathrm{R}) \sum_{k=1}^{K} \mathrm{Z}(t, t_{k-1}) \left[\mathrm{Q}(t, t_{k-1}) - \mathrm{Q}(t, t_k) \right]. \tag{3.25}$$

The distance between the bound scales has an error of $\dfrac{T-t}{K}$. A way to approximate this integral is by setting the value of the protection leg equal to the average of these bounds. So the present value of the protection is given by:

$$\begin{aligned} \text{Protection PV} &= \frac{1}{2}(\mathcal{L} + \mathcal{U}) \\ &= \frac{1 - \mathrm{R}}{2} \sum_{k=1}^{K} \left[\mathrm{Z}(t, t_k) + \mathrm{Z}(t, t_{k-1}) \right] \left[\mathrm{Q}(t, t_{k-1}) - \mathrm{Q}(t, t_k) \right]. \end{aligned}$$

Let us now analyze the error implied by this approximation. Writing $\mathrm{Z}(t, T) = \exp(-r(T - t))$ and $\mathrm{Q}(t, T) = \exp(-\lambda(T - t))$, putting $\epsilon = \dfrac{T-t}{K}$, and thinking in the first-order expansion in ϵ, it has been shown that the percentage of error is $\mathcal{O}(r(r - \lambda)\epsilon^2/12)$, with monthly time steps. The accuracy is in order of $\mathcal{O}(10^{-7})$, and we are in the required tolerance. In weekly time steps it falls to $\mathcal{O}(10^{-8})$, which is a great tolerance.

3.2.3 The Full Mark-to-Market

Now we are able to compute the full mark-to-market value of a long protection. The value of a long protection is the difference between the Protection Leg and the Premium Leg. So with a face value of

$\$1$, a contractual spread S_0, a maturity time T and a valuation date t, the value of the long protection position is given by

$$\mathrm{V}(t) = \frac{1-\mathrm{R}}{2}\sum_{k=1}^{K}\left[\mathrm{Z}(t,t_k)+\mathrm{Z}(t,t_{k-1})\right]\left[\mathrm{Q}(t,t_{k-1})-\mathrm{Q}(t,t_k)\right] - \mathrm{S}_0.\mathrm{RPV01}(t,T), \quad (3.26)$$

where

$$\begin{aligned}\mathrm{RPV01}(t,T) = &\sum_{n=1}^{N}\Delta(t_{n-1},t_n)\mathrm{Q}(t,t_n)\mathrm{Z}(t,t_n)\\ &+\frac{1}{2}\Delta(t_{n-1},t_n)\sum_{n=1}^{N}\mathrm{Z}(t,t_n)\left[\mathrm{Q}(t,t_{n-1})-\mathrm{Q}(t,t_n)\right].\end{aligned}$$

For a short position the valuation is just the negative value of this.

3.2.4 Breakeven Spread

The breakeven spread is determined upon a new contract. It is the credit default spread paid on it. We know that the value of a new contract in the initiation date at time $t = 0$ costs nothing, so we have $V(0) = 0$, using equation (3.26) and substituting for $t = 0$, we have

$$\mathrm{S}_0 = \frac{1-\mathrm{R}}{2}\frac{\sum_{k=1}^{K}\left[\mathrm{Z}(0,t_k)+\mathrm{Z}(0,t_{k-1})\right]\left[\mathrm{Q}(0,t_{k-1})-\mathrm{Q}(0,t_k)\right]}{\mathrm{RPV01}(0,T)}. \quad (3.27)$$

3.3 The Bootstrap Approach

After we explained how to value both protection and premium legs, we will now give more details on the pricing of a CDS contract. This can be priced given the issuer survival curve $\mathrm{Q}(t,T)$, a Libor curve $\mathrm{Z}(t,T)$, and an assumption about the expected recovery rate, R. In this section, we will build a survival curve that will reprice the full term structure of quoted CDS spreads, and finally this will allow us to value a CDS contract. The writer O'Kane[(10)], when building the survival curve, takes into consideration some properties that will be announced. They need to be established so that the survival curve fits to what it is believed to be the best approach accommodating the market prices.

The desirable properties are:

1. As we previously saw the minimum PV accuracy is in order of $\mathcal{O}(10^{-7})$, so the spread is defined with an error of $\mathcal{O}(10^{-4})$ basis points or less.

2. The method that we will present should interpolate between the market quotes in a sensible manner.

3. The construction method will be local. This means that if we are working with a 5Y CDS spread and rebuild the CDS curve it is preferable having a method which only changes the spread of CDS with a maturity close to 5Y.

4. The algorithm for the building curve should be fast.

5. The curve should be smooth, but we will prioritize localness to smothness.

With these properties, the curve construction approach will be based on the bootstrap approach.

3.3.1 Bootstrapping Approach Algorithm

The Bootstrap approach is one of the most stable curve construction and it is what we will use for constructing CDS survival curves.

As explained in O'Kane[10] a bootstrap works by starting with the shortest dated instrument and works out to the longest dated instrument, at each step using the price of the next instrument to solve for one parameter which determines how to extend the survival curve to the next maturity point. With this we have a survival curve which can replicate the market.

Since we need to use all the survival probabilities for all times between today and the CDS maturity, this bootstrap algorithm requires a linear interpolation scheme. For this interpolation scheme, we will use the log of the survival probabilities, since it is the quantity which satisfies all of the criteria that was announced earlier and it will ensure no-arbitrage as we will prove.

The log of the survival curve is obviously expressed in terms of the survival curve. Setting today to be time 0, we can rewrite the survival curve as

$$\mathrm{Q}(t) = \mathrm{Q}(0, T).$$

The interpolation scheme is posed as linear interpolation of some function $f(t)$ as shown is figure 3.3

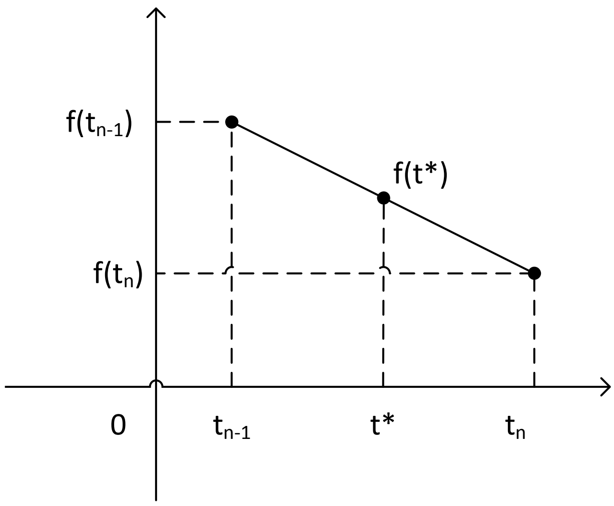

Figure 3.3: An interpolation scheme for $f(t)$.

Using this figure, we can have the standard linear interpolation formula to some time $t^* \in [t_{n-1}, t_n]$ as

$$f(t^*) = \frac{(t_n - t^*)f(t_{n-1}) + (t^* - t_{n-1})f(t_n)}{(t_n - t_{n-1})}. \tag{3.28}$$

To perform the interpolation, we introduce the continuously compounded forward default rate, $h(t)$, given by

$$\mathrm{Q}(t) = \exp\left(-\int_0^t h(s)\mathrm{ds}\right). \tag{3.29}$$

When the hazard rate process is deterministic, this equals the hazard rate. Doing $f(t) = -\ln \mathrm{Q}(t)$, and since $0 \leq \mathrm{Q}(t) \leq 1$, the minus sign ensures that $f(t)$ is a positive number.

Using equation (3.29) we can write

$$f(t) = \int_0^t h(s)\mathrm{ds}. \tag{3.30}$$

Then we have

$$h(t) = \frac{\partial f(t)}{\partial t}. \tag{3.31}$$

We then need to differentiate equation (3.28), so we can write this interpolation scheme to some time t^* in terms of $f(t)$. Hence,

$$h(t^*) = \frac{\partial f(t^*)}{\partial t^*} = \frac{f(t_n) - f(t_{n-1})}{t_n - t_{n-1}}. \tag{3.32}$$

We can see that $h(t^*)$ is constant in terms of t^*, so we can argue that $h(t)$ is constant between the interpolation limits. Therefore, the linear interpolation of the log the survival probability is equivalent to assuming a piecewise constant forward default rate $h(t)$.

We can define the constant continuously compounded forward default probability at time t^*, with $t_{n-1} < t^* < t_n$, as

$$\begin{aligned} h(t^*) &= \frac{1}{(t_n - t_{n-1})} \times (-\ln(\mathrm{Q}(t_n)) + \ln(\mathrm{Q}(t_{n-1}))) \\ &= \frac{1}{(t_n - t_{n-1})} \times \ln\left(\frac{\mathrm{Q}(t_{n-1})}{\mathrm{Q}(t_n)}\right). \end{aligned}$$

After doing some arrangements to have the formula for a survival probability to some time t^*, we get the following equation

$$\mathrm{Q}(t^*) = \mathrm{Q}(t_{n-1}) \exp(-(t^* - t_{n-1})h(t_{n-1})). \tag{3.33}$$

No-arbitrage requires that $h(t) \geq 0$. Thinking that this is the case at each of the skeleton points, which is the case if $\mathrm{Q}(t_n) \leq \mathrm{Q}(t_{n-1})$, this piecewise constant interpolation of $h(t)$ will also ensure no-arbitrage between these points such like we have mentioned before.

Algorithm

Now we will start to implement our model. We wish to build the survival curve using the curve building approach known as bootstrapping.

We define the set of CDS market quotes as $S_1, S_2, \cdots, S_M$ which are for contracts with times to maturity $T_1, T_2, \cdots, T_M$.

We need a recovery rate assumption R which we assume is the same for all maturities. The goal of the bootstrap algorithm is to produce a vector of survival probabilities $\mathrm{Q}(T_m)$ at the $M+1$ times (which includes 0) and reprice the CDS market quotes given the chosen interpolation scheme.

We also need to extrapolate the survival curve below the shortest maturity CDS and beyond the longest maturity CDS. So between $t=0$ and $t=T_1$ it is assumed that the forward default rate is flat at a level of $h(0)$. And beyond the last time point T_M it is assumed that the forward default rate is flat at its last interpolated value.

So the algorithm used by O'Kane[(10)] to build the survival curve is as follows:

1. Initialize the survival curve with $\mathbb{Q}(T_0 = 0) = 1.0$.
2. Set $m = 1$.
3. Solve for the value of $\mathbb{Q}(T_m)$ for which the mark-to-market value of the T_m maturity CDS with market spread S_m is equal to zero. Using equation
$$\mathrm{S}_0 = \frac{1-\mathrm{R}}{2}\frac{\sum_{k=1}^{K}(\mathrm{Z}(0,t_k)+\mathrm{Z}(0,t_{k-1}))(\mathrm{Q}(0,t_{k-1})-\mathrm{Q}(0,t_k))}{\mathrm{RPV01}(0,T)}, \tag{3.34}$$
all of the discount factors required to determine the CDS mark-to-market will be interpolated form the values of $\mathbb{Q}(T_1), \cdots, \mathbb{Q}(T_{m-1})$ which have already been determined, and $\mathbb{Q}(T_m)$ which is the value we are solving for.
4. Finding the value of $\mathbb{Q}(T_m)$ which reprices the CDS with maturity (T_m), we add this time and value to our survival curve.
5. Set $m = m+1$. If $m \leq M$ return to step (3).
6. We have now the survival curve of $M+1$ points with times at $0, T_1, T_2, \cdots, T_M$ and values $1.0, \mathbb{Q}(T_1), \mathbb{Q}(T_2), \cdots, \mathbb{Q}(T_M)$.

CDS Valuation

To highlight the implementation of the model we will now replicate the example of the credit default swap valuation illustrated by O'Kane (2008). To accomplish this task, it was necessary to construct a Libor Curve to calibrate the survival curve of the term structure of CDS spreads, and finally to use both of these curves to price a CDS contract.

We will value an existing CDS position with the following trade details:

As we can see, the contract has 6 years to maturity, and at the valuation date the deal has approximately 4.8 years remaining to maturity. Since the next coupon is on February, we are part way through a premium accrual period. The CDS swap curve and the Libor rates are the following:

Position	Short Position
Face Value	$ 10 million
Valuation Date	18 January 2008
Effective Date	15 November 2006
Maturity Date	15 November 2012
Contractual Spread	180 bp
Business day convention	Modified following

Libor Curve		**CDS Curve**	
6M deposit	4.650%	6M	145bp
1Y swap	5.020%	1Y	145bp
2Y swap	5.019%	2Y	160bp
3Y swap	5.008%	3Y	175bp
4Y swap	5.002%	4Y	190bp
5Y swap	5.030%	5Y	220bp
7Y swap	5.041%	7Y	245bp
10Y swap	5.080%	10Y	270bp

The author have assumed a 40% recovery rate and use these market rates to construct the survival CDS curve. The table 3.1 shows the requirements needed to calculate both the CDS curve as well for pricing the CDS.

The risky PV01 is equal to 4.2390, and takes into account the full accrued interest that will be received at the end of the current premium payment period. The mark-to-market is negative because at the time of the contract the five year swap spread was 180 bp and now is given by 220 bp, and it is a short position. The accrued interests equals $\$32,000$, corresponding to the time difference between the previous coupon (15 November 2007) and the valuation date. With a new contract to be started at the time of the valuation date, the new spread is given by 177.2 bp. As we can see in the figure 3.4 the hazard rate is steeply raising, following what we have explained in Chapter 2. The results reported in table 3.1 were obtained by implementing the market model of O'Kane and Turnbull[11] in Matlab. The code is available in Appendix A.

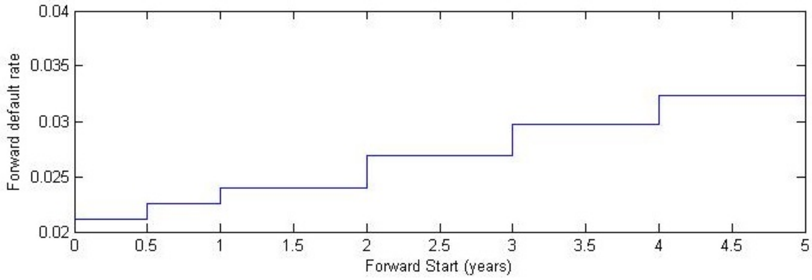

Figure 3.4: Behavior of the Hazard Rate

Table 3.1: CDS Valuation

Coupon Dates	Accrual Factor	Libor Zero Rates	Libor Discount Factor	Results	Survival probability in the paper	% Diference
18-Jan-08			1.000000	1.00000	1.00000	0
15-Feb-08	0.25556	4.380%	0.996605	0.99836	0.99741	-0.10%
15-May-08	0.25000	4.496%	0.985477	0.99311	0.99005	-0.31%
15-Aug-08	0.25556	4.712%	0.973250	0.98911	0.98325	-0.59%
17-Nov-08	0.26111	4.905%	0.960228	0.98329	0.97634	-0.71%
16-Feb-09	0.25278	5.032%	0.947776	0.97630	0.96970	-0.68%
15-May-09	0.24444	5.061%	0.936654	0.97059	0.96332	-0.75%
17-Aug-09	0.26111	5.092%	0.924774	0.96453	0.95656	-0.83%
16-Nov-09	0.25278	5.123%	0.913274	0.95869	0.95005	-0.90%
15-Feb-10	0.25278	5.153%	0.902038	0.94777	0.94359	-0.44%
17-May-10	0.25278	5.183%	0.891157	0.94135	0.93591	-0.58%
16-Aug-10	0.25278	5.214%	0.880275	0.93497	0.92759	-0.79%
15-Nov-10	0.25278	5.244%	0.869394	0.92863	0.91933	-1.00%
15-Feb-11	0.25556	5.276%	0.858645	0.91468	0.91106	-0.39%
16-May-11	0.25000	5.308%	0.848398	0.90791	0.90222	-0.63%
15-Aug-11	0.25278	5.341%	0.838038	0.90112	0.89290	-0.91%
15-Nov-11	0.25556	5.374%	0.827564	0.89430	0.88356	-1.20%
15-Feb-12	0.25556	5.413%	0.817177	0.87888	0.87433	-0.52%
15-May-12	0.25000	5.456%	0.807091	0.87181	0.86239	-1.08%
15-Aug-12	0.25556	5.499%	0.796781	0.86464	0.84853	-1.86%
15-Nov-12	0.25556	5.543%	0.786471	0.85754	0.83489	-2.64%
Risky PV01		4.2390		**Accrued Interest**		+32,000
Full Mark-to-Market		-181,482.25		**Clean mark-to-market**		-149,482.25
Full Mark-to-Market		-181,482.25		**Breakeven Spread**		177.2 bp

The differences between the 5th and the 6th columns of table 3.1 can be explained by the discount factors calculated and in the bootstrap method implemented. The discount factors are computed by bootstrapping the Libor rates above and then using a linear interpolation for the coupon dates.

We also calculate the survival probability using a built-in function of Matlab (*cdsbootstrap*) which is based in the same model we are explaining. This built in function was used first as an auxiliary function to build the model.

As we can show in table 3.2, if we assume that the Matlab function is the correct one, our model is more precise for the first 2 years and then the survival curve begin to show some significance differences. Once again, the discount factors are an essential factor for the calculation of the survival curve.

To conclude, I would like to show a little example for comparing the model that was explained and another model, i.e. the Hull and White[(6)] model. The difference between this model and the model that was explained before is that in the Hull and White model the price of the CDS is computed with the information about the bond that is insuring by the CDS.

A VBA code for such model is available in Löffler and N. Posch[(8)]. Implementing the model explained early, the results are shown in table 3.3 and the print of the excel where the formulas of the

Table 3.2: Differences in the model

Dates	Results of Matlab	Survival probability O'Kane	Results
18-Jan-08	1	1	1
15-Fev-08	0.99809	0.99741	0.99836
16-Fev-09	0.97402	0.96970	0.97630
15-Fev-10	0.94525	0.94359	0.94777
15-Fev-11	0.91224	0.91106	0.91468
15-Fev-12	0.87525	0.87433	0.87888

Dates	Dif Matlab vs O'Kane	Dif Model vs O'Kane	Dif Matlab vs Model
18-Jan-08	0.00%	0.00%	0.00%
15-Fev-08	0.07%	0.10%	0.03%
16-Fev-09	0.44%	0.68%	0.23%
15-Fev-10	0.18%	0.44%	0.27%
15-Fev-11	0.13%	0.40%	0.27%
15-Fev-12	0.11%	0.52%	0.41%

VBA code were implemented is in Appendix A, as the information of the bond for the inputs.

Considering what is showned in Appendix A, the probability of surviving per year accordingly to

Table 3.3: Results of the O'Kane model implemented

Dates	Survival Probability	Hazard Rate
21-11-2014	1	0.1458
20-10-2020	0.9888	0.138
21-11-2015	0.9763	0.1218
21-11-2016	0.9477	0.0984
21-11-2017	0.9146	0.0981
21-11-2018	0.8788	0.0912
21-11-2019	0.8293	0.0871

the model implemented by Hull and White[(6)] is 84.959% $(100\% - 15.041\%)$. With this probability the credit default swap spread is calculated giving us a CDS spread of 19.058%, as shown in Appendix A.

We also calculate this spread and the survival probabilities per year using the model explained by O'Kane and Turnbull[(11)]. Using the table 3.3, the survival probability on the maturity date of the contract is 82.93%, which is not very different from the Hull and White[(6)] model. But, when calculating the CDS spread, via O'Kane and Turnbull[(11)] this is given by 4.94%. The CDS in question is available in figure 3.5. The price of the CDS is given by 5.16716%, so we can conclude that our replication of the model implemented by O'Kane and Turnbull[(11)], accomodates better, than the model used by Hull and White[(6)] the results of bloomberg for the Novo Banco CDS.

But, we recall, that the model used by Hull and White[(6)] admits no counterparty risks where as in the model implemented, this risk is incorporated in the protection leg of the CDS.

Figure 3.5: Novo Banco CDS Euro Senior 5 Year

4

Conclusions

Conclusion

This work implements the Credit Default Swap (CDS) Valuation model proposed by O'Kane and Turnbull[11]. To pricing this CDS was necessary to study all the fundamental principles of stochastic calculus and financial derivates available in the Hull[5]. After implementing the algorithm in Matlab there are some differences between the algorithm and the results of O'Kane they can be explained by not only the differences in the discount factors but also the algorithm used by the author which we cannot access. Hence we have used also a built-in- function given by the Matlab, and with the same inputs, both the hazard rate and the survival probabilities were closer to our results.

With this kind of work, it is possible to know the probability of default of a certain company, just with a simple CDS contract. This is very useful nowaday. This model is a simple method, which does not use too many inputs, only the CDS in question, the Libor Rates and the CDS curve.

Many extensions could be performed to correctly price a CDS, depending if it is a covered or a subordinated bond (Cooper and Mello[2]). In this case we have ignored this characteristic, but this is one that can be considered by the CDS buyer.

For future work i would like to study other models to improve the pricing of a CDS and implementing such models in the valuation of the default probability of a company, but now with more inputs of the bond secured. My other future objective is to study how the market uses this instrument for hedging concentrations of credit risk, and study how the Options on CDS are made and their objective in the market.

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A

Appendix

```
%% Valuing the Survival Probability of the CDS%%%
% In this function the objective is calculate the root if the Auxiliar
% function
%% Sett: Settlemente of the CDS
%% Mat: Maturity of the CDS;
%% Va: Valuation Date;
%% Freq:Coupon Frequency;
%% Zero Data: Matrix with the information of the Zero Discount Factor for all maturities.
%% Spots: Libor Swap Curve;
%% Rates: CDS swap rates;
%% Rec: Recovery Rate;
%% T: maturity of the CDS;
%% h: hazard rate associated to maturity T-1.

function [a,Prob]= CDSValuation(Sett,Mat,Val,Freq,ZeroData,Spots,Rates,basis,Rec,T,h)

% Setting the erro tolerance
options = optimset('Display','off','TolX',1e-12,'TolFun',1e-12);
x0 = 0;
Val1=Val;
Listah=[];
% Set of dates corresponding to the coupons since the CDS has survive in the first period,
    then the valuation is the date of previous survival.
[b,t1]=PaymentDates(Val,Mat,Freq,basis);
if T>1
Val=datestr(t1((T-1)*4,1));
else
    Val=Val;
end
% Hazard rate that solves the equation
hazardr= fsolve(@(x) AuxFun(x,Sett,Mat,Val,Val1,Rates,Freq,ZeroData,Spots,Rec,basis,T,h), x0
    , options);
a=hazardr;
end

function y=AuxFun(x,Sett,Mat,Val,Val1,Rates,Freq,ZeroData,Spots,Rec,basis,T,h)
val=datenum(Val);
valu=datenum(Val1);
set=datenum(Sett);
mat=datenum(Mat);

%t - coupon dates in string type
%t1- coupon dates in num type

[t,t1]=PaymentDates(Val,Mat,Freq,basis);
[tl,t1l]=PaymentDates(Val1,Mat,Freq,basis);

if T>1
% Value of the coupon next to the date found
CA=(T-1)*5;
else
    CA=5;
end
if T==1
% b - dates corresponding to a coupon paying monthly frequency in string type
% t2 - dates corresponding to a coupon paying monthly frequency in num type
[b,t2]=PaymentDates(datestr(datenum(Val)-30),datestr(t1(CA,1)),12,basis);
else if T<=4
[b,t2]=PaymentDates(datestr(t1(1,1)-30/360),datestr(t1(5,1)),12,basis);
```

```
    else T>4
[b,t2]=PaymentDates(datestr(t1(1,1)-30/360),datestr(t1(4,1)),12,basis);
    end
end
  [bl,t2l]=PaymentDates(Val1,Mat,12,basis);

RPV01=0;
S=0;

for j=1:12*(T+1)+1
    tau(j,1)=(j-1)/12;
end


%%
%Calculate the Protection Leg, this will be done by discritising the time
%between coupons
M=12;
if T==1
    for k=1:M
%          Libor discount factor intial and final
      ri=1/(1+((t2(k,1)-val)/365)*LiborRate(Spots,(t2(k,1)-val)/365));%first discount factor
      Lista(k,1)=ri;

      rf=1/(1+((t2(k+1,1)-val)/365)*LiborRate(Spots,(t2(k+1,1)-val)/365));%secound discount factor
       Lista(k+1,1)=rf;

    S=S + (ri+rf)*(exp(-x*tau(k,1))-exp(-x*tau(k+1,1)));
    end
else
% Implementing the Bootstrapp approach
    S=0;
    if T>4
        M=9;
    else
        M=12;
    end
for k=1:M

    % For the next maturities the Libor Discount factors where calculated by bootstrapping
%first discount factor
       ri=LiborRate(ZeroData,(t2l((T-1)*12+k,1)-valu)/360);
       Lista(k,1)=ri;

     %secound discount factor
        rf=LiborRate(ZeroData,(t2l((T-1)*12+(k+1),1)-valu)/360);
       Lista(k+1,1)=rf;

%  Once found the previous Survival probability then the next is
% calculated using the previous hazard rate and so on to the final maturity
if T==2
    S=S + (ri+rf)*(exp(-h(1,1)-x*(tau(k,1)))-exp(-h(1,1)-x*(tau(k+1,1))));

else if T==3
    S=S + (ri+rf)*(exp(-h(1,1)-h(1,2)-x*(tau(k,1)))-exp(-h(1,1)-h(1,2)-x*(tau(k+1,1))));
    else if T==4
    S=S + (ri+rf)*(exp(-h(1,1)-h(1,2)-h(1,3)-x*(tau(k,1)))-exp(-h(1,1)-h(1,2)-h(1,3)-x*(tau(k+1,1))));
```

```
            else
                S=S + (ri+rf)*(exp(-h(1,1)-h(1,2)-h(1,3)-h(1,4)-x*(tau(k,1)))-exp(-h(1,1)-h(1,2)-h(1,3)-h(1,4)-x*(tau(k+1,1))));
            end
        end

end
end

end

%% Calculate The Risky Premium Present Value, which is calculated between coupon dates
% For the calculation of the risky premium present value is necessary to know the survival probability in coupon dates

for j=1:CA+2
%We need to consider all de time intervals
    a(j,1)=tau((j-1)*3+1,1);
end
ri=0;

   if T==1
       for j=1:T*4
  %Libor Discount Factor

    ri=1/(1+a(j+1,1)*LiborRate(Spots,a(j+1,1)));
r01(j,1)=ri;
     RPV01=RPV01+((a(j+1,1)-a(j,1))*(exp(-x*a(j,1))+exp(-x*a(j+1,1)))*ri);
       end

   else if T>4
%              In each year for a frequency of 4, exists 4 coupon dates, so
%              we need to calculate the Risky Premium Present Value in all of these coupon dates.
   Quart=3;
        else
   Quart=4;
        end
   r01=0;
    for j=1:Quart

          ri=LiborRate(ZeroData,(t1l((T-1)*4+j,1)-valu)/360);
          r01(j,1)=ri;
     if T==2

% As said before, once found the first survival probability this will be used to calculate th next one

     RPV01=RPV01+(((t1(j+1,1)-t1(j,1))/360)*(exp(-h(1,1)*(a(j,1)+1)-x*(a(j,1)))+exp(-h(1,1)*(a(j+1,1)+1)-x*(a(j+1,1))))*ri);
     else if T==3
     RPV01=RPV01+(((t1(j+1,1)-t1(j,1))/360)*(exp(-h(1,1)-h(1,2)-x*(a(j,1)))+exp(-h(1,1)-h(1,2)-x*(a(j+1,1))))*ri);
          else if T==4

     RPV01=RPV01+(((t1(j+1,1)-t1(j,1))/360)*(exp(-h(1,1)-h(1,2)-h(1,3)-x*(a(j,1)))+exp(-h(1,1)-h(1,2)-h(1,3)-x*(a(j+1,1))))*ri);
              else
```

```
      RPV01=RPV01+(((t1(j+1,1)-t1(j,1))/360)*(exp(-h(1,1)-h(1,2)-h(1,3)-h(1,4)-x*(a(j,1)))+
     exp(-h(1,1)-h(1,2)-h(1,3)-h(1,4)-x*(a(j+1,1)))))*ri);
                end
            end
        end


     end
        r01;

    end

Lista;
r01;
y=((1-Rec)*0.5*S/(RPV01/2))-Rates(1,T);
end
```

Table A.1: Inputs for the implementation of the Hull and White Model

	Coupon	Yield	Price	t	SpotRate
Corporate	1.73%	14.75%	32.97	0.01918	0.001%
Risk-free	1.73%		112.52	0.03836	-0.002%
				0.08219	0.009%
Maturity	27-05-2018			0.25479	0.041%
Coupon Freq	2			0.25205	0.081%
Recovery	40%			0.49863	0.181%
				0.5	0.181%
Settlement	27-05-2008			1.0	0.334%
Default freq	4			2.0	0.212%
				3.0	0.264%
				4.0	0.335%
				5.0	0.417%
				6.0	0.447%

Table A.2: Results of Hull and White Model

	Price of the bond	%Acrued Interest	Spot Rate	Loss	PD per year	Fees	Default Payments
27-May-08	112.52	0.8625%	0.00%	72.52	3.636%	24.09%	2.1688%
27-Aug-08	115.29	0.4313%	0.04%	75.30	3.636%	23.18%	2.1748%
27-Nov-08	115.29	0.8625%	0.18%	75.33	3.636%	22.25%	2.1668%
27-Feb-09	115.29	0.4313%	0.18%	75.35	3.636%	21.34%	2.1721%
27-May-09	115.29	0.8625%	0.18%	75.37	3.636%	20.42%	2.1648%
27-Aug-09	115.29	0.4313%	0.18%	75.38	3.636%	19.50%	2.1701%
27-Nov-09	115.29	0.8625%	0.18%	75.40	3.636%	18.59%	2.1629%
27-Feb-10	115.29	0.4313%	0.18%	75.42	3.636%	17.67%	2.1682%
27-May-10	115.29	0.8625%	0.18%	75.44	3.636%	16.76%	2.1609%
27-Aug-10	115.29	0.4313%	0.18%	75.46	3.636%	15.85%	2.1662%
27-Nov-10	115.29	0.8625%	0.18%	75.47	3.636%	14.93%	2.1590%
27-Feb-11	115.29	0.4313%	0.18%	75.49	3.636%	14.02%	2.1642%
27-May-11	115.29	0.8625%	0.18%	75.51	3.636%	13.11%	2.1570%
27-Aug-11	115.29	0.4313%	0.18%	75.53	3.636%	12.20%	2.1623%
27-Nov-11	115.29	0.8625%	0.18%	75.55	3.636%	11.30%	2.1551%
27-Feb-12	115.29	0.4313%	0.18%	75.56	3.636%	10.39%	2.1603%
27-May-12	115.29	0.8625%	0.18%	75.58	3.636%	9.48%	2.1531%
27-Aug-12	115.29	0.4313%	0.18%	75.60	3.636%	8.57%	2.1584%
27-Nov-12	115.29	0.8625%	0.18%	75.62	3.636%	7.67%	2.1512%
27-Feb-13	115.29	0.4313%	0.18%	75.64	3.636%	6.76%	2.1564%
27-May-13	115.29	0.8625%	0.18%	75.65	3.636%	5.86%	2.1492%
27-Aug-13	115.29	0.4313%	0.18%	75.67	3.636%	4.96%	2.1545%
27-Nov-13	115.29	0.8625%	0.18%	75.69	3.636%	4.06%	2.1473%
27-Feb-14	115.29	0.4313%	0.18%	75.71	3.636%	3.15%	2.1525%
27-May-14	115.29	0.8625%	0.18%	75.73	3.636%	2.25%	2.1453%
27-Aug-14	115.29	0.4313%	0.18%	75.74	3.636%	1.35%	2.1506%
27-Nov-14	115.29	0.8625%	0.18%	75.76	3.636%	0.45%	2.1434%
27-Feb-15	115.29	0.4313%	0.18%	75.78	3.636%	-0.44%	2.1486%
27-May-15	115.29	0.8625%	0.18%	75.80	3.636%	-1.34%	2.1415%
PD per period	3.760%		**PD p.a.**	15.041%		**CDS spread**	19.058%

Figure A.1: Bond covered by the CDS Sr 5Y